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| Basic Principle of Counting | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| How many diff mobile phone nums in SG? (1st digit either 8 or 9) | | | | | | | | | | | | | | | | | | | | | | | | 2 x 10 x 10 x 10 x 10 x 10 x 10 x 10 | | | | | | | | | | | | | |
| Num of 7-place license plates if 1st 2 letters are SF? | | | | | | | | | | | | | | | | | | | | | | | | 26 x 10 x 10 x 10 x 10 | | | | | | | | | | | | | |
| Num of committee of 1 Chinese, 1 M, 1 I from 6C, 4M, 2I | | | | | | | | | | | | | | | | | | | | | | | | 6 x 4 x 2 | | | | | | | | | | | | | |
| Permutations or Arrangement | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Form 4-digit num using 1357 | | 4! | | | Form 4-digit num using 1355 | | | | | | | | | | | | | 4!/2! | | | | | | Form 8-digit num using 23445555 | | | | | | | | | | 8! / (2!4!) | | | |
| 4M, 6W seated in a row if 4M must sit tgt | | | | | | | 7!4! | | | | | | | 2M, 4W seated in a row if 2M must not sit tgt | | | | | | | | | | | | | | | | | 6!-5!2! OR 4!2! | | | | | | |
| Arrangements of 123455 if 2 5's not tgt | | | | | 6!/2! | | | | | | Select r objs from set of n if order of selection is impt | | | | | | | | | | | | | | | | n x (n-1) x... x (n-r+1) = | | | | | | | | | | |
| Combinations | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Committee of 2 from 4 men | | | | Num of ways to select 2M if order impt = 4 x 3  Num of ways to select 2M if order not impt = (4x3)/2! | | | | | | | | | | | | | | | | | | | | = | | | | | | | | | | | | | |
| Select 6 num from 1,2,..., 45 | | | |  | | | | | | | | 5-card poker hands | | | | | | |  | | | | | Divide 4 men into 2 teams of 2 each | | | | | | | | | | | / 2! | | |
| Form committees of 3M, 2W from 6M, 5W | | | | | | | |  | | | | | Form committees of 3M, 2W from 6M, 5W if 1 of the M must be in committee | | | | | | | | | | | | | | | | | | | | | | | |  |
| Form committees of 3M, 2W from 6M, 5W if 2 of the W cannot serve tgt | | | | | | | | | | | | | | | | | | | | | | | | + | | | | | | | | | | | | | |
| Form committees of 3M, 2W from 6M, 5W if 2 of the W must be in committee if 1 of them is in | | | | | | | | | | | | | | | | | | | | | | | | + | | | | | | | | | | | | | |
| 4M, 3W seated in a row if no 2 women tgt | | | | | | | 4!3! | | | | | | | 4 black, 3 white marble in a row if no 2W are tgt | | | | | | | | | | | | | |  | | | | | | | | | |
| Expand (x + y)4 | x0y4 + x1y3 + x2y2 + x3y + x4y0 | | | | | | | | | | | | | | | | Num of subsets of a set consisting of n elems | | | | | | | | | | | | | = 2n | | | | | | | |
| Binomial Theorem.    #Let i = k+1 and let i = k | | | | | Proof. By MI, n = 1: LHS = x + y. RHS = x0y1 + x1y0 = y + x. Suppose result true for n-1.  Then (x + y)n = (x+y)(x+y)n-1 = (x + y) = + = + = xn + + yn + = xn + yn + = xn + yn + = | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Multinomial Coefficients | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Divide 9 ppl into 3 grps of size 2,3,4 | | | = | | | | | | | | | | Divide 9 ppl into 3 eql grps A,B and C | | | | | | | | | | | |  | Divide 9 ppl into 3 eql grps | | | | | | | | | | () / 3! | |
| Expand (x1 + x2 + x3)2 | | | | | | | | | | | | | | | + + + x1x2 + x1x3 + x2x3 | | | | | | | | | | | | | | | | | | | | | | |
| Num of Integer solns of Eqns | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Num of distinct +ve integer valued vectors (x1, x2, x3, x4) satisfying x1 + x2 + x3 + x4 = 9 | | | | | | | | | | | | | | | | | | | | | | | |  | | | | | | | | | | | | | |
| Divide 10 scouts into 5 teams A, B, C, D, E | | | | | | 510 | | | | Divide 10 scouts into 5 teams A, B, C, D, E if each team must receive 2 scouts | | | | | | | | | | | | | | | | | | | | | | = | | | | | |
| Divide 10 marbles into 5 boxes A, B, C, D, E | | | | | | | |  | | | | | | | Divide 10 marbles into 5 boxes A, B, C, D, E if each box ≥ 1 marble | | | | | | | | | | | | | | | | | | | |  | | |
| Extra | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Committee of 6 ppl from 7M, 8W. Need ≥ 3W, ≥2M. | | | | | | | | | | | | | | | | | | | | | | | | + | | | | | | | | | | | | | |
| Diff linear arrangement of letters A,B,C for which A is before B | | | | | | | | | | | | | | | | | | | | | | | | 3!/2 | | | | | | | | | | | | | |
| 8 ppl seated in a row if there are 4M, 4W and no 2M or 2W can sit tgt | | | | | | | | | | | | | | | | | | | | | | | | 4!4! x 2 | | | | | | | | | | | | | |
| 8 ppl seated in a row if there are 4 married couple and each couple must sit tgt | | | | | | | | | | | | | | | | | | | | | | | | 4!2!2!2!2! | | | | | | | | | | | | | |
| 5 awards given to class of 30. How many outcomes if student can receive any num of awards? | | | | | | | | | | | | | | | | | | | | | | | | 305 | | | | | | | | | | | | | |
| 5 awards given to class of 30. How many outcomes if student can receive ≤ 1 award? | | | | | | | | | | | | | | | | | | | | | | | | 30 x 29 x 28 x 27 x 26 | | | | | | | | | | | | | |
| Num of vectors (x1, ..., xn) s.t. xi is either 0 or 1 and ≥ k | | | | | | | | | | | | | | | | | | For = k, vector must have k 1's and n-k 0's, num of such vector =  For ≥ k, + +... + | | | | | | | | | | | | | | | | | | | |
| Num of vectors (x1, ..., xn) s.t. 1 ≤ xi ≤ n and x1 < x2 < ... < xk | | | | | | | | | | | | | | | | | | | | (choose k distinct integers for xi and then only 1 way to arrange it) | | | | | | | | | | | | | | | | | |
| Select committee of any size and chairperson for committee from n ppl | | | | | SUM (Select committee of size k and chairperson from this k ppl) =  select chairperson from n ppl, remaining n-1 ppl either in or not in committee | | | | | | | | | | | | | | | | | | | | | | | | | | | = n x 2n-1 | | | | | |
| Select committee of any size and chairperson and secretary (can same person as chairperson) for committee from n ppl | | | | | | | | | | | | | | | | | | | | | | | | = n x 2n-1 + n(n-1)2n-2 = n(n+1)2n-2 | | | | | | | | | | | | | |
| Committee has 8 members. How many way to choose p, t, s if member can take one role only. | | | | | | | | | 8 x 7 x 6 | | | | Num of way to choose p, t, s if A and B must serve tgt if either one selected. Member can take one role only | | | | | | | | | | | | | | | | 6 x 5 x 4 + 6 x 3! OR 3! + 6 x 3! | | | | | | | | |
| Arrangements of A, B, C, D, E, F, F if A must be btw 2 F's | | | | | | | | | | | | | | | | | | | | | | | | 5! | | | | | | | | | | | | | |
| 1st experiment can result in m diff outcomes. Suppose 1st experiment has outcome i, 2nd experiment can result in ni diff outcomes. Total outcomes of 2 experiments? | | | | | | | | | | | | | | | | | | | | | | | |  | | | | | | | | | | | | | |
| 10 W, 12M. If 5M, 5W chosen and paired off, how many diff pairs | | | | | | | | | | | | | | | | 5! | | | | | | 20 ppl shake hands. How many handshakes | | | | | | | | | | |  | | | | |
| Start from point A to go to point B. Only can move up or right. How many paths from A to B  Go from A to B but must pass through certain point at 2U2R from A | | | | | | | | | | | | | | | | | | | | | | | | (total 7 moves of up and right, 3 right) | | | | | | | | | | | | | |
| Prove = + + ... +  Let m = r = n and = = | | | | | | | | | | | | | | | | | | | | | | | There are grps of size r. There are grps of size r consisting of i men and r-i women | | | | | | | | | | | | | | |
| 8 teachers divided among 4 schools, how many divisions possible? | | | | | | | | | | | | | | | | 48 | | | | | What if each sch need 2 teachers? | | | | | | | | |  | | | | | | | |
| 3 weightlifters from US, 4 from Russia, 2 from China, 1 from Canada. If only considering countries of lifters, how many diff outcome possible. | | | | | | | | | | | | | | | | |  | | | | | | | How many outcomes if US has 1 in top 3 and 2 in bottom 3? | | | | | | | | | |  | | | |

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| Sample Space | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Diff types of sample spaces | | | | | | S = {abcd: a,b,c,d = 0,1,...,9} (4D draw) | | | | | | | | | | | | | | | | S = {x: 0≤ x < ∞} (lifespan) | | | | | | | | | | | | | S = {(H,H), (H,T), (T,H), (T,T)} (coin tossed 2 times) | | | | | | | | | | | |
| DeMorgan's Laws Proof. i. =  Proof. Let x x x E1 and x E2 and ... x En  x and x and ... x x  Let x x and x and ... x x E1 and x E2 and ... x En x x . Thus = | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | ii. =  Proof. Using 1st law of DeMorgan,  = = (since (EC)C = E)  Thus, = | | | | | | | | | | | | | | |
| Some simple propositions | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 60% of students wear neither ring nor necklace. 20% wear ring and 30% wear necklace. Prob that a student wearing ring or necklace, P(RN)  Prob that student wearing ring and necklace, P(RN) | | | | | | | | | | | | | | | | | | | | | | | | | | | | Let R = {ring}, N = {necklace}. P((RN)C) = 0.6, P(R) = 0.2, P(R) = 0.3  P(RN) = 1 - P((RN)C) = 0.4  P(RN) = P(R) + P(N) - P(RN). P(RN) = 0.2 + 0.3 - 0.4 | | | | | | | | | | | | | | | | | | |
| 1. P() = 0 | | | | | | | | | | Proof. Let E1 = S, Ei = for i > 1, then S = and Ei are mutually exclusive.  Using axiom 3, P = . P(S) = P(S) + P(E2) + P(E3) + ... = P(S) + P() + P() + ... P() = 0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2. P = (when sample space is finite) | | | | | | | | | | | | | | | | | | | Proof. Let Ei = fpr i > n and apply axiom 3 | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 7. P(EC) = 1 - P(E) | | | | | | | | | | | | | | Proof. 1 = P(S) = P(EEC) = P(E) + P(EC) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 8. If E F, then P(E) ≤ P(F) | | | | | | | | | | | | | | Proof. Let F = EECF. P(F) = P(E) + P(ECF). Since P(ECF) ≥ 0,... | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 9. P(EF) = P(E) + P(F) - P(EF) | | | | | | | | | | | | | | Proof. Let EF = EECF. P(EF) = P(E) + P(ECF) – (1). F = EF ECF. P(F) = P(EF) + P(ECF) – (2)  Substitute 2 into 1, P(EF) = P(E) + P(F) - P(EF) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 11. Inclusion-exclusion identity Proof P(E1E2...En) =    –  + ...  + (-1)r+1  + ...  + (-1)n+1P(E1E2...En) | | | | | | | | | | | | | | If outcome is not in any Ei, then its prob don't contribute to either side of eqn  Suppose outcome is in m of events Ei, m > 0, let prob of outcome = w, then  i. outcome is in E1E2...En and w will be counted once on LHS  ii. outcome is in m of Ei and w will be counted times in  iii. outcome is in subsets of and w counted times in and so on...  w = w – w + w – ... ± w = 1 = – + – ... ±  = 0 (always true, by using binomial theorem and let x = -1, y = 1) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 12i. P ≤ Proof.  Note P(Ei) = P() + P()  P() = P(Ei) – P() – sub this result in | | | | | | | | | | | | | | | | | | | | = E1 ...  P = P(E1) + P() + P() +...+ P()  = P(E1) + = –  Since ≥ 0, thus P ≤ | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 12ii. P ≥ – Proof.  Note = [P(E2E1)  + P(E3E1) + P(E3E2) + ...  + P(EnE1) + P(EnE2) + ... + P(EnEn-1)] = | | | | | | | | | | | | | | | | | | | | | | = P(EiE1 EiE2 ... EiEi-1) =  P() ≤ = (from 9i.)  P = – (from proof of 9i.)  ≥ – = – | | | | | | | | | | | | | | | | | | | | | | | | |
| Sample Spaces having equally likely outcomes | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Prob sum of 2 dice = 4 | | | | | 3/36 | | Select 8 chips from 10 defective and 90 non-defective. Prob ≤ 1 chip is defective | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | + | | | | | |
| Form committee of 5 from 6M and 9W. P(A), Prob John, 2M and 2W are selected?  P(B), Prob 3M (no John) and 2W selected? | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | P(A) = P(B) = | | | | | | | | | | | | | |
| Suppose 3 red and 2 blue balls arranged s.t. all 5! possible orderings are equally likely. If we now record result by listing colors of balls, show that all possible results remain equally likely. | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | total arrrangements of rrrbb = , prob = 1/ = | | | | | | | | | | | | | |
| A box contains n balls of which 1 is special. If k balls are withdrawn 1 at a time, prob that special ball is chosen? | | | | | | | | | Mtd 1: prob = = = | | | | | | | | | | | | | | | | | Mtd 2: Let Ai = event that special ball = ith ball chosen, i = 1,2,...,k  Since any of n balls is equally likely to be ith ball chosen, P(Ai) = 1/n  OR Total num of outcomes = n(n-1)(n-2)...(n-k+1) =  Num of outcomes of A­i = (n-1)(n-2)...(n-i+1)(1)(n-i)...(n-k+1) =  P(Ai) = [] / [] = = 1/n  P({special ball chosen}) = P() = = k/n | | | | | | | | | | | | | | | | | | | | |
| P(full house) = | P(straight) = (A2345, 23456,...,10JQKA) x (4 suits5 - 4 outcomes with all same suit(straight flush)) | | | | | | | | | | | | | | | | | | | | | | P(4 of a kind) = | | | | | | | | | | | | | | P(flush) =  4 suits x (any 5 cards - 10 possible straight) | | | | | | | | | |
| P(2 pairs) = = (since 2 grps) | | | | | | | | | | | | | | | | | | | | | | | P(1 pair) = | | | | | | | | | | | | | | | P(3 of a kind) = | | | | | | | | |
| birthday problem. If n ppl are in a room, prob that no 2 of them have the same b.d. | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | . If n ≥ 23, prob is less than 1/2 | | | | | | | | | | | | | |
| Cards are turned up 1 at a time until 1st ace appears. Is the next card more likely to be ace of spades or 2 clubs? | | | | | | | | | | | | | | | | | | P(ace) = = (arrange the 51 cards 1st: 51!. Then put ace spades after 1st ace: only 1 way)  P(2 clubs) = 1/52 also | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Football team has 20 offensive and 20 defensive players. Players are paired in grps of 2. | | | | | | | | | | | | | P(no O-D pair) = | | | | | | | | P(2 O-D pair) = , choose 2 O, 2 D, then 2! to arrange these grps | | | | | | | | | | | | | | | | | | | | | | | | | |
| Suppose N men randomly select a hat. | | | Prob that none select their own hat?  Let Ei = event ith man select own hat, i = 1,2,...,N  P(≥ 1 man select own hat) = P = – + ... + (-1)r+1 + ... + (-1)N+1P(E1E2...EN) (point 11)  For event , let the r men (i1, i2,..., ir) select own hats first, other (N-r) men have (N-r)! ways of selecting remaining (N-r) hats  =  = = , where = num of terms  P = 1 - + - ... + (-1)N+1  P = 1 - 1 + - + ... + (-1)N e-1 ≈ 0.37 as N ∞ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | Prob exactly k men select their own hat?  num of ways only k men select own hats =  N choose k men \* [P(N - k don't select own hat) \* total num of ways N - k men select hat] =  [1 - 1 + - + ... + (-1)N-k ](N - k)!  P(only k men select own hat) = = = e-1/k! as N ∞ | | | | | | | | | | | | | |
| Prob as a cts set fn | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Incr seq: E1 E2 E3 ... En En+1... Hence =  Decr seq: E1 E2 ... En En+1 ... And =  If {En, n ≥ 1} is either incr or decr seq, then = P() | | | | | | | | | | | | | | | | | | | | Proof. Suppose {En, n ≥ 1} is incr seq. Let F1 = E1, F2 = E2...  Fn = En = En since En-1 =  Note Fn are mutually exclusive events and = n ≥ 1  P() = P() = = = = = | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Extra | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Deck of 20 cards numbered 1,2,3,..., 20 shuffled. Card selected 1 at a time without replacement until all 20 selected. | | | | | | | | | | | P(3rd card is 9) = 1/20 OR 19!/20!  P(1st 3 cards all odd num) = OR | | | | | | | | | | | | | | | | | | | | | | | | P(last 3 cards all odd num) = | | | | | | | | | | | |
| Roll 5 dice. | | P(no two same) = | | | | | | P(1 pair) = | | | | | | | | | P(2 pair) = | | | | | | | | P(3 same) = | | | | | | | | P(full house) = | | | | | | | P(4 same) = | | P(5 same) = | | | | |
| P(neither you nor dealer has blackjack). Let A = {you have blackjack}, B = {dealer has blackjack}. Ans = 1- P(AB) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | P(AB) = P(A) + P(B) - P(AB) = 2 x - | | | | | | | | | | | | | | |
| P(2nd die higher value than 1st) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | 1 = P(1st is higher) + P(2nd is higher) + P(same)  1 = 2 x P(2nd is higher) + 6/36  P(2nd is higher) = 5/12 | | | | | | | | | | | | | |
| Woman has n keys, of which 1 will open door. Tries opening door, discarding those that don't work, P(open on kth try) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | P(open on kth try) = =  P(open on kth try without discarding key) = | | | | | | | | | | | | | |
| Num of ppl in room for P(at least 2 of them have same birthday month) to be ≥ 0.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | P(all months diff) = . When n = 5, this prob < 0.5 | | | | | | | | | | | | | | | |
| Closet contains 10 pairs of shoes. If 8 shoes randomly selected, | | | | | | | | | | | P(no complete pair) = | | | | | | | | | | | | | | | | | | | | | | P(exactly 1 complete pair) = | | | | | | | | | | | | | |
| =  Let x x or x x E1 and x E2 and .... or x F x E1 or x F and x E2 or x F and .... x  Let x x E1 and x E2 and .... x E1 and x E2 and .... or x F x or x F x | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | = can be proven similarly | | | | | | | | | | | | | |
| P(EFC ECF) = P(EFC) + P(ECF) (disjoint events). P(E) = P(EFC EF) = P(EFC) + P(EF) and P(F) = P(ECF) + P(EF)  P(EFC ECF) = P(E) - P(EF) + P(F) - P(EF) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Consider matching problem, Let AN = num of ways N men can select hats so no man select own  Let BN = num of ways N men can select among N hats that does not contain the hat of 1 of these men | | | | | | | | | | | | | | | | A­N = (N-1)BN-1 BN-2 = AN-1/(N-2)  AN = (N-1)[AN-2 + (N-2)BN-2] (N-1: num of ways 1st man select a hat not his own, AN-2: extra man choose hat of 1st man, N-2: num of ways extra man choose hat that is not the 1st man hat  AN = (N-1)[AN-2 + AN-1] | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Parallel sys with n components. P(component work) = 0.5. P(component 1 works|sys working) | | | | | | | | | | | | | | | | P(component 1 works|sys working) = = | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 freshman boys, 6 freshman girls, 6 sophomore boys. How many sophomore girls for sex and class to be indep when student selected at random? | | | | | | | | | | | | | | | | P(Boy, F) = , P(Boy) = , P(F) = . For independence, P(Boy, F) = P(Boy)P(F)  = . 4x = 36 and x = 9. | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Stock price moves up by 1 unit with prob p, down 1 unit with prob 1-p. P(original price after 2 days)  P(up by 1 after 3 days)  P(up on 1st day|up by 1 after 3 days) | | | | | | | | | | | | | | P(original price after 2 days) = p(1-p) + (1-p)p = 2p(1-p).  P(up by 1 after 3 days) = p2(1-p) = 3p2(1-p)  P(up on 1st day|up by 1 after 3 days) = = = | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| P(heads) = p. P(HHHH), P(THHH), P(THHH before HHHH) | | | | | | | | | | | | | | | | P(HHHH) = p4. P(THHH) = (1-p)p3. P(THHH before HHHH) = 1 - P(HHHH) (since HHHH can only appear in 1st 4 flips) = 1 - p4 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Suppose E and F are mutually exclusive events. If indep trials are performed, show E will occur before F with prob P(E)/[P(E)+P(F)] | | | | | | | | | | | | | | | | P(E before F) = P(E before F|E 1st)P(E 1st) + P(E before F|F 1st)P(F 1st) + P(E before F|E and F not 1st)(1-P(E 1st)-P(F 1st) = P(E 1st) + P(E before F)(1-P(E 1st)-P(F 1st)) = P(E) + P(E b f|1-P(E)-P(F). So P(E b F) = P(E)/[P(E)+P(F)] | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Die A has 4r, 2w faces. Die B has 2r, 4w faces. Fair coin is flipped. If head, die A is used, else die B. Show P(red) = 0.5. P(r on 3rd|1st 2 r). P(A|1st 2 r) | | | | | | | | | | | | | | | | P(red) = (1/2)(4/6) + (1/2)(2/6) = 1/2. P(r on 3rd|1st 2 r) = = =  P(A|rr) = = = | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Let S = {1,2,..., n}. Suppose A and B are independently, equally likely to be any of the 2n subsets (including null set and S itself) of S.  Show P(AB) = (3/4)n. Show P(AB = ) = (3/4)n | | | | | | | | | | | | | | | | P(AB) = = = = (2+1)n =  P(AB = ) = P(ABC) = since BC also equally likely to be any of the subsets. | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Person is declared guilty if at least 2 out of 3 judges vote guilty. Suppose person is in fact guilty, each judge will vote guilty w prob 0.7. If person actually innocent, prob vote guilty = 0.2. If 70% of people are guilty, P(judge 3 vote guilty|j1 and 2 vote guilty)  P(j3 vote guilty|j1 and 2: 1 guilty and not guilty vote)  P(j3 vote guilty|j1 and 2 both vote not guilty)  Let Ei be event judge i cast guilty. Are events indep? Are they conditionally indep? | | | | | | | | | | | | | | | | P(judge 3 vote guilty|j1 and 2 vote guilty) = =  P(j3 vote guilty|j1 and 2: 1 guilty and not guilty vote) = =  P(j3 vote guilty|j1 and 2 both vote not guilty) = =  P(E1) = P(E2) = 0.7(0.7) + 0.3(0.2) = 0.55  P(E1E2) = 0.7(0.7)2 + 0.3(0.2)2 = 0.355 ≠ P(E1)P(E2). Similarly, all 3 events are not indep.  P(E1E2E3|guilty) = P(E1|guilty)P(E2|guilty)P(E3|guilty)  P(E1E2|guilty) = P(E1|guilty)P(E2|guilty)... show for E1E3 and E2E3, so conditionally indep. | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Prove if E1, E2, ..., En are indep, then P(E1E2...En) = 1 - | | | | | | | | | | | | | | | | | | | | | | | | P(E1E2...En) = 1 - P = 1 - | | | | | | | | | | | | | | | | | | | | | | |
| Fair coin tossed 2 times. A = {1st toss heads}. B = {2nd toss head}. C = {both toss on same side}. Show A, B, C are pairwise indep, but not indep. | | | | | | | | | | | | | | | | | | | | | | | P(A) = P(B) = P(C) = 1/2. P(AB) = P(AC) = P(BC) = 1/4. So P(AB) = P(A)P(B)...  However, P(ABC) = 1/4 ≠ P(A)P(B)P(C). So pairwise indep, but not indep | | | | | | | | | | | | | | | | | | | | | | | |
| If 0 ≤ ai ≤ 1, i = 1,2,..., show + = 1 | | | | | | | | | | | | | | | | Suppose want to calculate prob of flipping coin until head appears. = P(1st head on ith flip). = P(all tails). | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Weather tmr will be same as weather tdy with prob p. If weather is dry on Jan 1, show Pn = prob dry n days later satisfies Pn = (2p-1)Pn-1 + (1-p), P0 = 1. Prove Pn = 1/2 + 1/2(2p-1)n | | | | | | | | | | | | | | | | | | | | | Pn = pPn-1 + (1-p)(1-Pn-1) = (2p-1)Pn-1 + (1-p).  Using MI, suppose Pn-1 is true. Pn = (2p-1)Pn-1 + (1-p) = (2p-1)[1/2 + (1/2)(2p-1)n-1] + 1-p = (2p-1)/2 + (2p-1)n/2 + 1-p = 1/2 + (1/2)(2p-1)n | | | | | | | | | | | | | | | | | | | | | | | | | |
| Chessboard has 64 squares. Prob placing 8 rooks wont all be in same row or col. | | | | | | | | | | | | | | |  | | | | | | | | | | | | P(blackjack) | | | | | | | (order matters) OR (order dont matter) | | | | | | | | | | | | |
| 2 symmetric dice both have 2 sides R, 2B, 1Y, 1W. P(both same color) | | | | | | | | | | | | | | | | | | | | | | | P(RR) + P(BB) + P(YY) + P(WW) = + + + | | | | | | | | | | | | | | | | | | | | | | | |
| 20 families. 4 have 1 child. 8 have 2 child. 5 have 3 child. 2 have 4 child. 1 has 5 child. Total 48 children | | | | | | | | | | | | | | | | | | P(child chosen come from family with i children)?, i = 1,2,3,4,5. Let B = no of children in child family. P(B = 1) = 4/48, P(B = 2) = 16/48, P(B = 3) = 15/48, P(B = 4) = 8/48, P(B = 5) = 5/48 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 sch chess club contain 8, 9 players. 4 members from each club randomly chosen to join contest. Random player from 1 team paired with those from other team. Suppose R and E are in diff sch. | | | | | | | | | | | | | | | | | | | | | P(R and E paired) = = . P(R and E chosen but not paired) = - =  P(exactly one of R and E chosen) = | | | | | | | | | | | | | | | | | | | | | | | | | |
| Urn contains 3R, 7B balls. A and B select balls consecutively w/o replacement until R ball chosen | | | | | | | | | | | | | | | | | | | | | P(A select R) = | | | | | | | | | | | | | | | | | | | | | | | | | |
| Forest has 20 elk, of which 5 are captured, tagged and then released. 4 elk then captured. | | | | | | | | | | | | P(2 of these 4 are tagged) = | | | | | | | | | | | | | | | | | | Bridge: 13 cards, Yarborough: bridge with no cards higher than 9 (exclude ace) | | | | | | | | | | | | |  | | | |
| Teacher gives class 10 qns. Test contain 5 of those qns. If student knows how to ans 7 qns. | | | | | | | | | | | | P(all 5 qns correct) = . P(at least 4 correct) = + | | | | | | | | | | | | | | | | | | | | | | | | | | | Die roll 4 times. P(6 appears at least once)? | | | | | 1 - | | |
| n socks, 3R in drawer. Value of n if when 2 socks chosen, P(RR) = .5 | | | | | | | | | | | | | | | | | | | | | = .5. n = 4 | | | | | | | | 5 hotels. If 3 ppl check into hotels in a day, P(all diff hotel)? | | | | | | | | | | | | | | | | |  |
| A B C D E arranged in linear order. | | | | P(1 person btw A and B) = . P(2 ppl btw) = . P(3 ppl btw) = 3!2! | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | 6M, 6W divided into 2 groups of 6 each. P(both grp same num of men)? | | | | | | | | |  | |
| P(bridge is void of at least 1 suit). S: void of spade, H, D, C... | | | | | | | | | | | | | | | | | | P(SHDC) = [P(S) + P(H) + P(D) + P(C)] - [P(SH) + P(SD) + P(SC) + P(HD) + P(HC) + P(DC)] + [P(SHD) + P(SHC) + P(SDC) + P(HDC)] - P(SHDC) = - + - 0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

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| Conditional Prob and Reduced Sample Space | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Draw cards one at a time without replacement from deck. Let A = {1st card = ace}, B = {2nd card = ace} | | | | | | | | | | | | | | | | | | | | | | | | | P(A) = 4/52. P(B|A) = 3/51. P(B) = 4/52 | | |
| Toss fair die twice. Given 1st die is 5, what is conditional prob that sum of 2 dice is even? | | | | | | | | | | | | | | | | | | | | P(E|F) = 3/6 (since 1st already 5, 6 outcomes for 2nd die)  P(E) = 18/36 (consider whole sample space for unconditional prob) | | | | | | | |
| Toss fair die twice. Given 1st die is num < 3, prob sum of 2 dice is > 7 | | | | | | | | | | | | | | | | | | | | | | | | P(E|F) = 1/12 (only 1 possible outcome; (2,6))/  (1st die 1: 6 outcomes + 1st die 2: 6 outcomes) | | | |
| Take 8 balls from 4 black and 6 white balls sequentially. Given 5 of the 8 chosen are white, prob that 1st ball chosen is white? | | | | P(1st ball W|5 of 8 balls are W) = 5/8 (since 1st ball equally likely to be any of 8 chosen balls) | | | | | | | | | | | | | | | | OR Let B = {1st ball chosen W}. B5 = {5 W balls chosen}. P(B) = . P(B5) = . P(B5|B) = . P(B|B5) = = = | | | | | | | |
| P(first 2 cards are aces). Let A = {1st card ace}, B = {2nd card ace} | | | | | | | | | | | | | | | | | | | | | | | | P(BA) = P(A)P(B|A) = x | | | |
| Divide 52 cards into 4 piles of 13 each. Prob each pile has exactly one ace?  Note P(E4) = P(E1E2E3E4) | | Let E1 = {ace spades in any pile}, E2 = {ace spades, ace hearts in diff pile}, E3 = {ace spades, hearts diamonds diff pile}, E4 = {all aces diff pile}  P(E1) = 1. P(E2|E1) = . P(E3|E1E2) = . P(E4|E1E2E3) = P(E1E2E3E4) = P(E1)P(E2|E­1)P(E3|E1E2)P(E4|E1E2E3) = 0.105 | | | | | | | | | | | | | | | | | | | | | | | | OR P(E4) = = 0.105 | |
| Multiplication rule: P(E1E2...En) = P(E1) P(E2|E1)P(E3|E1E2)...P(En|E1E2...En-1) | | | | | | | | | | | | | | | | | | Proof. RHS = P(E1) x x x... x = P(E1E2...En) | | | | | | | | | |
| Thrm of Total Prob and Bayes' Thrm | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Blood Test is 95% TP. 1% FP.  6% of pop has disease | | P(test +ve) = 0.06 x 0.95 + (1-0.06) x 0.01 = 0.0664 | | | | | | | | | | | | | | | | | | | | P(disease|test +ve) = (0.06 x 0.95) / 0.0664 = 0.06  P(no disease|test -ve) = (1-0.06) x (1-0.01) / (1-0.664) = 0.94 | | | | | |
| Urn 1 initially has n red balls and urn 2 has n blue balls. Remove 1 ball from urn 1, take 1 ball from urn 2 and put in urn 1. Repeat until all molecules removed from both urn.  Let R = {last ball removed from urn 1 is red}  If urn 1 now has r1 red balls and b1 blue balls, urn 2 has r2 red balls and b2 blue balls. Find P(R) | | | | | | Let 1 of the n red balls be a special one. F = {s ball is final one selected}. Ni = {s ball not ith ball removed}  P(N1) = . P(N2|N1) = ... P(F|N1N2...Nn) = . Note F N1N2...Nn. P(F) = P(N1N2...NnF) =  P(N1)P(N2|N1)... P(Nn|N1N2...Nn-1)P(F|N1N2...Nn) (multiplication rule) = x ... x =  Since special ball can be any of n red balls, P(R) = P(F) x n = e-1 as n ∞ | | | | | | | | | | | | | | | | | | | | | |
| Let 1 of the balls in urn 1 be special one. Now P(F) = P(N1N2...F) = P(N1)P(N2|N1)... P(|N1N2... )P(F|N1N2... ) = x... x x =  Let O = {last ball removed is one of the ball originally in urn 1}  P(O) = x () =  P(R) = P(R|O)P(O) + P(R|OC)P(OC) = x + x | | | | | | | | | | | | | | | | | | | | | |
| Plant A, B, C produce 20%, 30%, 50% of RAM chips. % of defective chips by A, B and C are 1%, 2%, 4% respectively. | | | | | | | | | | | | | | | | | | | | | | | | P(defective) = 0.2(0.01) + 0.3(0.02) + 0.5(0.04) = 0.028  P(plant A|defective) = 0.2(0.01)/0.028 = 0.071 | | | |
| Couple with 2 children. Suppose we encounter mother walking with one of her child. If child is a girl, prob that both are girls?  So actually prob depends on how mom choose child to walk with her  If mom only chose elder child, then P(G|G1B2) = 1, P(G|B1G2) = 0 and P(G1G2|G) =  If mom only choose girl, then P(G|G1B2) = 1, P(G|B1G2) = 1 and P(G1G2|G) = | | | | | | | | | | | | | | | | | | | | Let G1 = {1st child girl}, G2 = {2nd child girl}, G = {child seen with mom is girl}...for boys also...  P(G1G2|G) = = since G1G2 G  P(G) = 0.25 \* 1 + 0.25 \* P(G|G1B2) + 0.25 \* P(G|B1G2) + 0.25 \* 0  P(G1G2|G) = = | | | | | | | |
| Conditioning formula: P(E) = P(E|F)P(F) + P(E|FC)P(FC) | | | | | | | | | | | | | | | | | | | | Proof. E = EFC EF. P(E) = P(EFC) + P(EF) = P(E|FC)P(FC) + P(E|F)P(F) | | | | | | | |
| Independent Events | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Draw cards from 52 cards without replacement. C = {2nd card is ace}. A = {1st card ace}  C = {2nd card is ace}. A = {1st card red color}  C = {2nd card is ace}. B = {1st card diamond} | | | | | | | | | | | | | | | | | | | | | | | | P(C) = 4/52. P(C|A) = 3/51. So A and C not indep  P(C) = 4/52. P(C|A) = = P(C). A and C indep  P(C) = 4/52. P(C|B) = = P(C). B and C indep | | | |
| Toss 2 fair dice. A = {1st die 3}, B = {sum is 5}, C = {sum is 8}, D = {sum is 7}  P(D) = 6/36. P(D|A) = 1/6. A and D indep | | | | | | | | | | | | | | | | | | | | | | | | P(B) = 4/36 = 1/9. P(B|A) = 1/6. A and B not indep  P(C) = 5/36. P(C|A) = 1/6. A and C not indep | | | |
| E, F and G are indep E and F G are indep | | | | | | | | | | | | | | | | | | | | | | | | E.g. P(E(F G)) = P(EF EG) = P(EF) + P(EG) - P(EFG) =  P(E)P(F) + P(E)P(G) - P(E)P(F)P(G) ( indep) =  P(E)[P(F) + P(G) - P(F)P(G)] = P(E)P(F G) | | | |
| Seq of Bernoulli trials(Binomial). Infinite seq of indep. trials are performed. Success has prob p, failure: 1-p. | | | | | | | | | | | | | | | | | | | | | | | P(at least 1 success occur in 1st n trials)= 1 - (1-p)n  P(exactly k success occur in 1st n trials) = pk(1-p)n-k  P(all trials are success) = pppp... = = | | | | |
| Sys contains 3 components arranged in parallel. If component i functions with prob pi, i = 1,2,3 | | | | | | | | P(sys functions) = 1 - P(sys don't fn) (since need ≥ 1 component functioning) = 1 - (1-p1)(1-p2)(1-p3)  P(sys functions in series) = p1p2p3 | | | | | | | | | | | | | | | | | | | |
| Box contains 5 balls, 2 are W. A draw first, then B with replacement. Winner is 1st to draw W | | | | | P(A wins) = + + + ... = [1 + + + ...] = [1/(1-)] = | | | | | | | | | | | | | | | | | | P(A wins) = + P(A wins) (recursive solution)  P(A wins)[1-] = | | | | |
| Player A picks one of the following: HHH, HHT, HTH, HTT, THH, TTH, THT, TTT  Player B then picks one of the remaining 7 patters. A fair coin is tossed until either A or B pattern appears  If A picks HHH, B THH. P(A wins) = 1/8 (only 1 route). A picks HHT, B picks THH. P(A wins) = 1/4 (HHH or HHT)  A picks HTH, B picks HHT, P(B wins) = 2/3. P(B wins|HH) = .5P(B wins|HH) + .5(1). P(B wins|HH) = 1 – (1).  P(B wins|HT) = .5(0) + .5P(B wins|TT). P(B wins|TT) = 2P(B wins|HT) – (2)  P(B wins|TH) = .5P(B wins|HH) + .5P(B wins|HT) = .5 + .5P(B wins|HT) (from (1)). 2P(B wins|TH) = 1 + P(B wins|HT) – (3)  P(B wins|TT) = .5P(B wins|TH) + .5P(B wins|TT). P(B wins|TT) = P(B wins|TH) – (4)  Sub (4) into (3): 2P(B wins|TT) = 1 + P(B wins|HT) - (5)  Sub (2) into (5): 4P(B wins|HT) = 1 + P(B wins|HT). P(B wins) = 1/3 – (6)  Sub (6) into (5): P(B wins|TT) = 2/3 – (7). Sub (7) into (4): P(B wins|TH) = 2/3  P(B wins) = .25P(B wins|HH) + .25P(B wins|HT) + .25P(B wins|TH) + .25P(B wins|TT) = 2/3 | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
| Problem of the points - Fermat and Pascal. Independent trials, with Success prob p, failure: 1-p. Probability n success before m failures? | | | | | | | Pascal: Let Pn,m = P(n success before m failures)  Pn,m = pPn-1,m + (1-p)Pn,m-1, n ≥ 1, m ≥ 1  Pn,0 = 0. P0,m = 1. P1,1 = pP0,1 + (1-p)P1,0 = p | | | | | | | | | | | | | | | | | Femat: winner must emerge after m+n-1 trials  For n success before m failures, need at least n success in n+m-1 trials  Pn,m = | | | |
| Gambler's ruin problem. A with $M, B with $N. P(success) = p, P(failure) = 1-p. Success: B gives A $1. Failure: A gives B $1. Game stops when either A or B wins | | | | | | | | | | | | | | | | | | | Let Pi = P(someone wins all M+N starting with $i). P0 = 0, PM+N  = 1  Pi = 0.5Pi-1 + 0.5Pi+1 Pi+1 = 2Pi - Pi-1  P2 = 2P1 - P0 = 2P1. P3 = 2P2 - P1 = 3P1. Pi = 2Pi-1 - Pi-1 = iP1  So PM+N = (M+N)P1 = 1 P1 = 1/(M+N)  P(A wins) = PM = MP1 = M/(M+N). P(B wins) = PN = NP1 = N/(M+N) | | | | | | | | |
| E and F are independent P(E|F) = P(E) | | | P(E|F) = = if E and F are indep = P(E) | | | | | | | | | | | | | | | | | | | | | | | | |
| E and F indep E and FC indep | E = EF EFC. P(E) = P(EF) + P(EFC) = P(E)P(F) + P(EFC) (since E, F indep) P(EFC) = P(E) - P(E)P(F) = P(E)(1-P(F)) = P(E)P(FC) | | | | | | | | | | | | | | | | | | | | | | | | | | |
| P(E|F) is a prob | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Seq of Bernoulli trials. P(success) = p, P(failure) = 1-p. P(n consecutive success before m consecutive failures)?  Let E = {n consecutive successes before m consecutive failures}  H = {first trial is success}. F = {trials 2 to n all success}  G = {trials 2 to m all failures} | | | | | | | | | | | | | | | P(E) = P(H)P(E|H) + P(HC)P(E|HC) = pP(E|H) + (1-p)P(E|HC)  P(E|H) = P(E|FH)P(F|H) + P(E|FCH)P(FC|H)  = 1 \* pn-1 + P(E|HC)(1-pn-1) (since once fail, restart from beginning)  P(E|HC) = P(E|GHC)P(G|HC) + P(E|GCHC)P(GC|HC)  = 0(1-p)m-1 + P(E|H)[1-(1-p)m-1] –solve simultaneous eqn and sub into P(E) | | | | | | | | | | | | |
| k+1 coins in box. P(ith coin = heads) = i/k, i = 0,1,...,k. Coin randomly selected from box and then repeatedly flipper. P(n+1 flip = head|first n flips all head)  Let ci = {ith coin selected}. Fn = {1st n flips all heads}. H = {n+1 flip = head}  If k = 3, ≈ + + (use rectangle to estimate area) = | | | | | | | | | | | | | | | | | | | | | | P(H|Fn) = P(ci|Fn)  P(H|Fnci) = P(H|ci) = i/k  P(ci|Fn) = = = =  P(H|Fn) = = ≈ = = | | | | | |
| Extra | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Deck of cards numbered 1,2,..., 20 selected 1 at a time w/o replacement until all 20 selected. | | | | | | | | | | | | | | | | | P(3rd, 4th card are odd|1st, 2nd card are odd) = =  P(1st, 2nd card are odd|3rd, 4th card are odd) =  P(card numbered 10 is among last 5 cards|1st, 2nd card are odd) = = | | | | | | | | | | |
| Urn A contains 2W, 4R balls; urn B contains 8W, 4R balls; urn C contains 1W, 3R balls. 1 ball is selected from each urn. | | | | | | | | | | | | | P(A = W|2W) = = = = | | | | | | | | | | | | | | |
| Divide 52 cards into 4 piles of 13 each. Prob each pile has exactly one ace?Let Ei = event ith hand has exactly 1 ace. Find p = P(E1E2E3E4) | | | | | | | | | | | | | | | P(E1) = , P(E2|E1) = , P(E3|E1E2) = , P(E4|E1E2E3) = 1. p = P(E1)P(E2|E1)P(E3|E1E2)P(E4|E1E2E3) (by multiplication rule) | | | | | | | | | | | | |
| 48% of the women and 37% of the men of a class remained nonsmokers for at least 1 year. These ppl attended a success party. If 62% of original class were male, | | | | | | | | | | | | | | P(women|attending party) = = ≈ 0.443  P(attending party) = 0.48\*0.38 + 0.37\*0.62 = 0.4118 | | | | | | | | | | | | | |
| Urn I contains 2W, 4R balls, urn II contains 1W, 1R ball. Ball randomly chosen from urn I and put into urn II and ball then randomly selected from urn II. | | | | | | | | | P(ball selected from urn II is white) = P(W|W transferred) + P(W|W not transferred) = + =  P(W transferred|W selected) = = = 1/2 | | | | | | | | | | | | | | | | | | |
| 2 balls are either painted black or gold with prob = 0.5 and placed in a box. | | | | | | | | | | P(both gold|at least 1 gold) = =  If box tips over and 1 gold ball falls out. P(both gold) = 1/2 since no other info about ball in box. | | | | | | | | | | | | | | | | | |
| 5% of M and 0.25% of W are color blind. Assume there are equal num of M and F.  What is there were twice as many M as F | | | | | | | | | | | | | | | | | | | | | | P(M|color blind) = =  When p = 1/2, P(M|C) = 0.9524. If p = 2/3, P(M|C) = 0.9756 | | | | | |
| Deck of cards turned over 1 at a time until 1st ace appears. Let E = 1st ace is 20th card, A = {21st card is ace spades}, D = {21st card is ace}, C = {2 clubs in 1st 20 card}, B = {21st card is 2 clubs} | | | | | | | | | | | | | | | | | | | | | P(A|E) = P(D|E)P(A|DE) + P(DC|E)P(A|DCE) = + =  P(B|E) = P(C|E)P(B|CE) + P(CC|E)P(B|CCE) = (0) + = | | | | | | |
| 3 coins in a box. One is 2-headed coin; another is fair coin; third is biased coin with P(head) = 75%. | | | | | | | | | | | | | | | | | | | | | | P(2-headed coin|head) = = | | | | | |
| 10 coins, coin i P(head) = i/10, i = 1,2,..., 10. | | | | | | | | | | | | | | | | | | | | | | P(5th coin|head) = = | | | | | |
| 2 identical cabinets has 2 drawers. A contains a silver coin in each drawer, B contains silver coin in 1 drawer and gold coin in the other. | | | | | | | | | | | | | | | | | P(other drawer has silver coin|drawer has silver coin) = = 2/3 | | | | | | | | | | |
| Prob that good, ave, bad risk person would be in accident is 0.05, 0.15, 0.30. If 20% of pop is good, 50% is ave, 30% is bad, what proportion of ppl have accident? | | | | | | | | | | | | | | | | | | | | | | P(accident) = 0.2\*0.05 + 0.5\*0.15 + 0.3\*0.3 = 0.175  P(good|no accident) = . P(ave|no accident) = | | | | | |
| |  |  |  | | --- | --- | --- | | Day | P(mail|accepted) | P(mail|rejected) | | M | .15 | .05 | | T | .20 | .1 | | W | .25 | .1 | | Th | .15 | .15 | | F | .1 | .20 |   P(accepted) = .6 | | | | | | | | | | | P(M) = P(M|A)P(A) + P(M|R)P(R) = .15(.6) + .05(.4) = .11  P(T|MC) = =  P(A|MCTCWC) = = =  P(A|Th) = = =  P(A|no mail) = = = | | | | | | | | | | | | | | | | |
| Box 1 contains 2W, 3B balls. Box 2 contains 4W, 3B balls. Fair die is tossed, if num is 1 or 2, ball is randomly selected from box 1. Else, ball selected from box 2. | | | | | | | | | | | P(W) = + = . P(box 2|W) = = =  Same experiment carried out twice. P(both same color) = P(WW) + P(BB) = + = | | | | | | | | | | | | | | | | |
| 2 fair dice are rolled. P(one 6|dice lands on diff num)? | | | | | | | | | | | P(one 6|dice lands on diff num) = 10/(36-6) | | | | | | | | | | | | | | | | |
| P(1st die 6|sum = 7) = (1/36)/(6/36). P(1st die 6|sum = 8) = (1/36)/(5/36). P(1st die 6|sum = 9) = (1/36)/(4/36). P(1st die 6|sum = 10) = (1/36)/(3/36). P(1st die 6|sum = 11) = (1/36)/(2/36). P(1st die 6|sum = 12) = (1/36)/(1/36) | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 cards chosen without replacement. B = {both cards aces}. AS = {ace spades chosen}. A = {at least 1 ace chosen} | | | | | | | | | | | | P(B|AS) = = . P(B|A) = (since B subset of A) = = | | | | | | | | | | | | | | | |
| Urn contains 5W, 7B balls. Each time ball is selected, its color is noted and replaced in the urn along with 2 other balls of the same color. | | | | | | | | | | | | | | | | P(1st 2 balls are B and next 2 W) = \* \* \* . P(of the 1st 4 balls selected, only 2 are black) = P(WWBB, WBWB, WBBW, BWWB, BWBW, BBWW) | | | | | | | | | | | |
| 36% own dog. 22% of families that own dog also cat. 30% own cat. P(D) = 0.36. P(C|D) = 0.22. P(C) = 0.3. P(C) = P(CD) + P(CDC) = P(C|D)P(D) + P(C|DC)P(DC). P(C|DC) = (0.3-0.22\*0.36)/0.64 = 0.345 | | | | | | | | | | | | | | | P(random selection own dog and cat) = P(CD) = P(C|D)P(D) = 0.22\*0.36 = 0.0792  P(D|C) = = = = 0.264 | | | | | | | | | | | | |
| 46% independents. 30% liberals. 24% conservative. In an election, 35% of I, 62% of L, and 58% of C voted. | | | | | | | | | | | P(V) = 0.35\*0.46 + 0.62\*0.3 + 0.58\*0.24 = 0.4862. P(I|V) = 0.35\*0.46/0.4862 = 0.331. P(L|V) = 0.3\*0.62/0.4862 = 0.383. P(C|V) =0.24\*0.58/0.4862 = 0.286. | | | | | | | | | | | | | | | | |
| Let E = {1st ace is 20th card}. A = {21st card is ace of spade}. D = {20th card is ace of spades}. C = {2 of club among 1st 20 cards}. B = {21st card is 2 club} | | | | | | | | | | | P(A|E) = P(A|DE)P(D|E) + P(A|DCE)P(DC|E) = 0 + =  P(B|E) = P(B|CE)P(C|E) + P(B|CCE)P(CC|E) = 0\* + (29 position to place 2 clubs) | | | | | | | | | | | | | | | | |
| B hits target with prob p1. D hits target with prob p2. Suppose they shoot simultaneously at same target.  Assume both shots are indep | | | | | | | | | | | P(both hit|at least 1 hit) = =  P(B hit|at least 1 hit) = | | | | | | | | | | | | | | | | |
| Current score: B (87wins, 72lost), G (86, 73), D (86, 73)  G has 3 more games to play against D.  B has 3 more games to play againts P. (P cannot win division). Given prob of winning a game is .5. If 2 teams tie for 1st place, have additional game with same prob. | | | | | | | | | | | P(B wins 3) = 1/8. P(B wins 2) = 3(1/8) = 3/8. P(B wins 1) = 3/8. P(B wins 0) = 1/8  P(G wins 3) = 1/8 = P(G wins 0). P(G wins 2) = 3/8 = P(D wins 1)....  P(B wins division) = P(B wins 3) + P(B wins 2)[P(G wins 3)\*.5 + P(G wins 2) + P(G wins 1) + P(G wins 0)\*.5] + P(B wins 1)[P(G wins 2)\*.5 + P(G wins 1)\*.5] = 1/8 + 3/8\*[1/16+3/8+3/8+1/16] + 3/8[3/16+3/16] = 38/64 | | | | | | | | | | | | | | | | |
| Council contains 7 members, of which 3 members are in a steering committee. A legislation is first revied by committee members, and then by whole council if at least 2 of 3 committee approve. At council, at least 4 of 7 approve for legislation to pass. Prob of approval is p. P(given steering committee member vote is decisive)? Decisive if person vote is reversed, legislature reversed. | | | | | | | | | | | |  |  |  | | --- | --- | --- | | Given member | Other 2 member of steering committee | Other 4 council member | | for | both for | 3 against | | for | 1 for, 1 against | at least 2 for | | against | 1 for, 1 against | at least 2 for | | against | both for | 3 against |   P(decisive) = (p3)4p(1-p)3 + [p\*2p(1-p)][6p2(1-p)2 + 4p3(1-p) + p4] + [(1-p)\*2p(1-p)][6p2(1-p)2 + 4p3(1-p) + p4] + [(1-p)p2][4p(1-p)3] | | | | | | | | | | | | | | | | |
| Urn contains 12 balls, of which 4 are W. A, B, C draw from urn successively. Winner is 1st 1 to draw W ball. | | | | | | | | | | | If ball is put back after drawing. P(A wins) = + P(A wins). P(A wins) = .  If balls not replaced after drawing. P(A wins) = + + | | | | | | | | | | | | | | | | |
| Urn contains n W and m B balls. Balls withdrawn 1 at a time until only those of the same color are left. Show P(W left) is n/(n+m)  Pond contains R,B,G fish. r R, b B, g G fishes. Fish randomly removed. P(R fish 1st to be completely removed). | | | | | | | | | | | | Balls left all W if last ball drawn is W. Any of the n+m balls could be the last ball, so P = n/(n+m) | | | | | | | | | | | | | | | |
| P(R) = P(RBG) + P(RGB)  P(RBG) + P(G last)P(RBG|G last) = \* . P(RGB) = P(B last)P(RGB|B last) = \* | | | | | | | | | | | | | | | |
| If E1 and E2 are indep. then they are conditionally independent given F. Prove of give counterexample. | | | | | | | | | | | E1 and E2 indep ? P(E1E2|F) = P(E1|F)P(E2|F)  Suppose fair die is tossed twice, Let E1 = {1st toss 1}, E2 = {2nd toss 2}, F = {sum = 4}  P(E1E2|F) = 0. But P(E1|F) and P(E2|F) > 0. So statement is false | | | | | | | | | | | | | | | | |

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| Applications of r.v. | | | | | | | | | | | | | | | | | | | | | | | | | |
| John claims to have extrasensory perception. Card is randomly drawn from 4 cards. Test is done 10 times. If Joh gets 7 out of 10 correct, does he have ESP?  Let X = num of times out of 10 guess correctly | | | | | | | | | | | | | | | | | | | | | | X is binomial dist with n = 10, p = 0.25  P(X ≥ 7) = = 0.0035  Very unlikely, so John most likely has ESP | | | |
| Channel transmits digits 0 and 1. P(digit incorrectly received) = 0.2. To reduce error, 00000 is transmitted instead of 0 and 11111 instead of 1. Receivers uses 'majority' decoding, e.g. 1 = 11111 -> 10111 = 1 (true), 1 = 11111 -> 10100 = 0 (false). Let X = num of digits out of 5 that are received incorrectly. | | | | | | | | | | | | | | | | | | | X is binomial dist with n = 5 and p = 0.2  P(digit wrong when decoded) = P(X ≥ 3) = = 0.058  If only 1 digit was sent P(wrong) = 0.2, so improved accuracy | | | | | | |
| Coupon problem. There are N distinct types of coupons and selection is random. Let T = num of coupons that need to be collect until 1 obtain complete set of each type. Let Aj = event no type j coupon is obtained in first n coupons collected. P(T = n)?  Note P(T>n) = 1 if n < N  For 1 ≤ n ≤ N, = 1  Now, let Dn = num of distinct types of coupons obtained in 1st n selections. P(Dn = k)?. Let A = event coupon is one of these k types; B = event all k types appear at least once | | | | | | | | | | | | | | Consider P(T>n-1) = P(T≥n) = P(T=n) + P(T>n). P(T=n) = P(T>n-1) - P(T>n)  By inclusion-exclusion identity, P(T>n) = P( = – + ... + (-1)k+1 + ... + (-1)N+1P(A1A2...AN)  P(Aj) = (i.e. none of the n coupons are j)  = , = , 1P(A1A2...AN) = 0  P(T>n) = N - +... + (-1)N+ 0 =  P(A) = . P(B|A) = P(T ≤ n; from k types instead of N) = 1 -  P(Dn = k) = P(AB) = | | | | | | | | | | | |
| Pmf & Cdf | | | | | | | | | | | | | | | | | | | | | | | | | |
| |  |  |  |  | | --- | --- | --- | --- | | a | 1 | 2 | 4 | | P(X=a) | 0.5 | 0.25 | 0.25 |   Find cdf of discrete r.v. | | | | | | | | | | | | | | F(a) = | | | | | | | | E(X) = = 1(0.5) + 2(0.25) + 4(0.25) = 2 | | | OR E(X) = = P(X≥1) + P(X≥2) + P(X≥3) + P(X≥4) + P(X≥5) + ... = 1 + 0.5 + 0.25 + 0.25 + 0 +... = 2 |
| E(X) | | | | | | | | | | | | | | | | | | | | | | | | | |
| Find E(X) of indicator variable I. I = | | | | | | | | | | | | | | | | | | | | | | P(I = 1) = P(A). P(I = 0) = P(AC). E(I) = 1P(I = 1) + 0P(I = 0) = P(A) + 0 = P(A) | | | |
| E[g(X)] | | | | | | | | | | | | | | | | | | | | | | | | | |
| |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | x | -2 | -1 | 0 | 1 | 2 | | P(X=x) | 0.05 | 0.1 | 0.3 | 0.2 | 0.35 |   Let X be discrete r.v. with pmf:  Find E(X2) | | | | | | | | | | | | | | | | | | Let Y = X2. P(Y=0) = P(X=0) = 0.3  P(Y=1) = P(X=-1 or X=1) = 0.1+0.2 = 0.3  P(Y=4) = P(X=-2 or X=2) = 0.05+0.35 = 0.4  E(X2) = E(Y) = 0(0.3) + 1(0.3) + 4(0.4) = 1.9 | | | | | | | OR E(X2) = (-2)2(0.05) + (-1)2(0.1) + 02(0.3) +12(0.2) + 22(0.35) = 1.9 |
| Discrete r.v. E[g(X)] = | | | | | | Proof. = = = = E(g(X)) | | | | | | | | | | | | | | | | | | | |
| Variance | | | | | | | | | | | | | | | | | | | | | | | | | |
| Find Var(X) and Var(Y). P(X = 50) = 1. P(Y = 0) = P(Y = 100) = 0.5 | | | | | | | | | | | | | | | | | | | | Hence E(X) = E(Y) = 50. Var(X) = E(X2) - [E(X)]2 = 502(1) - 502 = 0  Var(Y) = E(Y2) - [E(Y)]2 = 02(0.5) + 1002(0.5) - 502 = 2500 | | | | | |
| Var(X) = E(X - )2 = E(X2) - [E(X)]2 | | | | | | | Proof. Var(X) = E(X - )2 = = = - 2 + = E(X2) - 2 + = E(X2) - = E(X2) - [E(X)]2 | | | | | | | | | | | | | | | | | | |
| Var(aX + b) = a2Var(X) | | | | | | | Proof. Var(aX+b) = E{[(aX+b) - E(aX+b)]2} = E{[(aX+b) - (a+b)]2} = E{a2(X-)2} = a2E(X-)2 = a2Var(X) | | | | | | | | | | | | | | | | | | |
| Applications | | | | | | | | | | | | | | | | | | | | | | | | | |
| Pepys problem. More likely to get at least an ace in 6 rolls of a die or at least 2 aces in 12 rolls of a die | | | | | | | | | | | | | | | | | | | | | | P(≥ 1 ace in 6 roll) = 1 - (5/6)6 = 0.67  P(≥ 2 ace in 12 rolls) = 1 - (5/6)12 - 12(1/6)(5/6)11 = 0.62 | | | |
| Multiple choice test contains 20 qns with 5 choices for each qn. If guess randomly...  Let X = num of correct ans, then X ~ Binomial(20, 0.2) | | | | | | | | | | | | | | | | | | | | | | P(X > 10) = = 0.0006  E(X) = 20(0.2) = 4 | | | |
| Person buys a particular 4D number 3 times a week. How long will the person need to strike 1st prize? Let X = num of trials until person strikes 1st prize. X ~ Geometric(1/10000) | | | | | | | | | | | | | | | | | | | | | | E(X) = 1/(1/10000) = 10000  10000/3 = 3333 weeks = 3333/52 = 64 years | | | |
| 10 ppl are tested for certain disease. Their blood samples are pooled and analysed tgt. If test = -ve, only 1 test required. If test = +ve, all 10 ppl need to be individually tested. Assume P(disease) = p, let T = num of test needed for 10 ppl | | | | | | | | | | | | | | | | | | | | | | P(T = 1) = (1-p)10. P(T = 11) = 1 - (1-p)10  E(T) = 1(1-p)10 + 11[1-(1-p)10]  If p = 0.5, E(T) = 10.99, (in this case dont pool blood tgt) | | | |
| St petersburg paradox. Toss a fair coin until tail appears. If tail appear on nth flip, then win $2n  Let X = winnings of a player. Need to pay $1000000 to play once. Let P = profit. P = X - 10000 | | | | | | | | | | | | | | | | | | | | | | E(X) = = = ∞  E(P) = E(X) - 10000000 = ∞ | | | |
| Coupon problem. How many coupons need to obtain a complete set? Suppose N = 8. Let X = num of coupons collected until complete set is obtained. Xi = num of additional coupons needed after i distinct types have been obtained in order to obtain another distinct type, i = 0,1,... 7. X = X0 + X1 + X2 + ... + X7 | | | | | | | | | | | | | | | | | | | | | | X0 ~ Geometric(1), X1 ~ Geometric(7/8), X2 ~ Geometric(6/8), ... X7 ~ Geometric(1/8)  E(X) = E(X0) + E(X1) + E(X2) + ... + E(X7) = 1 + 8/7 + 8/6 + ... + 8/1 = 21.7 | | | |
| If X ~ Binomial(n, p) and Y ~ Binomial(n-1, p), then E(Xk) = np \* E[(Y + 1)k-1]  Let k = 2, then E(X2) = npE[Y+1] = np[E(Y) + 1] = np[(n-1)(p) + 1] | | | | | | | | Proof. E(Xk) = = = = np = np = np = np \* E[(Y + 1)k-1] | | | | | | | | | | | | | | | | | |
| If X ~ Binomial(n,p), P(X = k) = P(X = k-1), k = 1,2,..., n | | | | | | | | | | | | | | | | Proof. = = | | | | | | | | | |
| Poisson r.v | | | | | | | | | | | | | | | | | | | | | | | | | |
| Errors on a page has Poisson dist w = 0.25. Let X = num of errors on given page. X ~ Poisson(0.25) | | | | | | | | | | | | | | | | | | | | | | | P(X ≥ 1) = 1 - P(X = 0) = 1 - e-0.25 = 0.22 | | |
| 500000 ppl spend $1 each on a 4D num of their choice. Expected num of ppl win. 1st prize? P(> N ppl win 1st prize)? Let X = num of ppl strike 1st prize. X ~ Binomial(500000, 1/10000) ≈ X ~ Poisson(500000/10000 = 50) | | | | | | | | | | | | | | | | | | | | | | | | E(X) = 50. P(X > N) = | |
| Hat matching problem. n men randomly select hat. P(none of the men select own hat)?  P(X = 0) ≈ e-1 ≈ 0.37 using inclusion-exclusion identity | | | | | | | | Note that X = + +... + , where IE = and Ei = ith man select own hat  P(Ei) = 1/n. P(Ei|Ej) = 1/(n-1) ≠ P(Ei). Thus E1, E2, ..., En not indep. But their dependence is weak for large n. So X ~ Poisson() where = nP(Ei) = 1. P(X = i) = = e-1/i!, i = 0,1,2,.... And P(X = 0) = e-1 | | | | | | | | | | | | | | | | | |
| Birthday problem. n ppl in a room. P(no two of them have same bd)  . If n ≥ 23, prob is less than 1/2 | | Consider pairs of person i and j, i ≠ j. Eij = {person i and j have same bd}  Let X = num of pairs with same bd = where =  P(Eij) = 1/365, P(Eij|Ejk) = 1/365. But P(Eij|EjkEik) = 1. So Eij only pair-wise indep but dependence is weak. So X~Poisson() where = = . P(X = i) ≈ . P(X = 0) = . And P(X = 0) ≤ 0.5 when n ≥ 23 | | | | | | | | | | | | | | | | | | | | | | | |
| Road accidents happen at rate of 5 per day. Let Y = num of accidents that occur in next 2 days. Y ~ Poisson(5\*2 = 10)  Let W = num of accidents in next 3 days. W ~Poisson(15)  Find dist of time starting from now, until next accident | | | | | | | | | | | | P(Y ≥ 10) = = 1 - . P(W = 0) = e-15  Let X = time (in days) until next accident. V = num of accidents in interval [0,t], V~Poisson(5t)  P(X > t) = P(no accident in interval [0,t]) = P(V = 0) = e-5t. P(X ≤ t) = 1 - e-5t | | | | | | | | | | | | | |
| Poisson approximation of binomial: If X ~ Binomial(n,p), n is large and p is small, then X~Poisson() approximately, where = np | | | Proof. P(X = i) = = = =  For large n, small p, ≈ 1, = 1()...( ≈ 1  = 1 + n + +... (Binomial theorem) = 1 - + +... = 1 - + + ... ≈ . P(X = i) ≈ | | | | | | | | | | | | | | | | | | | | | | |
| Binomial(n,p), n is large and p is small, then X~Poisson() approximately, where = np  E(X) = . Var(X) = | | | | | | | | Proof. E(X) = = = = =  E(X2) = = = = = [E(X + 1)] = ( + 1)  Var(X) = E(X2) - [E(X)]2 = ( + 1) - = | | | | | | | | | | | | | | | | | |
| Let N(t) = num of events occuring in time interval [0,t]. If the 3 assumptions are true, then N(t) ~ Poisson(t), where is rate of occurrences of events per unit time.  Divide timeline into n subintervals of length h, then nh = t  Note n = t + t t as n ∞  So, = for large n and  = 1 - | | | | | | | Proof. Let {N(t) = k} = A B. A = {k of n subintervals contain exactly 1 event and other n-k subintervals contain 0 events}. B = {N(t) = k and at least 1 subinterval contain ≥ 2 events}  P(B) = P({N(t) = k} {at least 1 subinterval contain ≥ 2 events}) ≤ P(at least 1 subinterval contain ≥ 2 events) = P() ≤ = (from assumption 2) = n = t 0 as n ∞ for a fixed t.  P(0 events in interval of length h) = 1 - [h + o(h)] - o(h) #from assumption 1 & 2 = 1 - h - o(h) # since o(h) + o(h) = o(h)  P(A) = . So this become like a binomial problem where n is large, so P(A) as n ∞. Thus, P(N(t) = k) = , k = 0,1,2,... | | | | | | | | | | | | | | | | | | |
| Expected Value of Sum of r.v. | | | | | | | | | | | | | | | | | | | | | | | | | |
| E(X) = = , where si = {s: X(s) = xi} | | | | | | | Proof. Suppose that distinct values of X are xi, i ≥ 1. For each i, let Si be event X = xi, i.e. Si = {s: X(s) = xi}  E(X) = = = = = | | | | | | | | | | | | | | | | | | |
| 2 indep flips of a coin with prob p. Let X = num of heads obtained.   |  |  |  |  | | --- | --- | --- | --- | | x | 0 | 1 | 2 | | P(X = x) | (1-p)2 | 2p(1-p) | p2 | | | | | | | | S = {(t,t), (h,t), (t,h), (h,h)}. X(t,t) = 0. P(X(t,t)) = (1-p)2. X(h,t) = 1. P(X(h,t)) = p(1-p).  X(t,h) = 1. P(X(t,h)) = (1-p)p. X(h,h) = 2. P(X(h,h)) = p2  E(X) = = 0(1-p)2 + 1(2p)(1-p) + 2p2  = 0(1-p)2 + 1p(1-p) + 1(1-p)p + 2p2 | | | | | | | | | | | | | | | | | | |
| For r.v. X1, X2, ..., Xn, E = | | | | | | | | | | | | | Let Z = = X1 + X2 + ... + Xn. E(Z) = = = + ... + = E(X1) + ... + E(Xn) | | | | | | | | | | | | |
| X~Binomial(n, p)  E(X) = np  Var(X) = np(1-p) | Let X = X1 + X2 + ... + Xn where xi = . P(success) = p. P(failure) = 1-p  E(Xi) = 1p + 0(1-p) = p. Then E(X) = E(X1) + E(X2) + ... + E(Xn) = p + p + ... + p = np  E(X2) = E = E = E + E = +  = E(Xi) = p ( = Xi). XiXj = . E(XiXj) = 1P(Xi = 1, Xj = 1) + 0 = 1P(Xi = 1)P(Xj = 1) ( indep) = p\*p = p2  E(X2) = + = np + (n2 - n)p2. Var(X) = E(X2) - [E(X)]2 = np + n(n-1)p2 - [np]2 = np(1-p) | | | | | | | | | | | | | | | | | | | | | | | | |
| Extra | | | | | | | | | | | | | | | | | | | | | | | | | |
| Given 10 pokeballs to catch pokemon. P(catch) = 0.2. Let X = num of balls to catch pokemon. X ~Geo(0.2) | | | | P(X = 3) = 0.82(0.2) = 0.128. P(X = 10|1st 7 balls fail to catch) = P(X = 3) = 0.128. (since balls are thrown independently). P(catch within 10 balls) = 1 - 0.810 ≈ 0.89. | | | | | | | | | | | | | | | | | | | | | |
| 2 balls chosen randomly from urn containing 8W, 4B, 2O balls. Suppose we win $2 for each B ball selected and lose $1 for each W ball selected. Let X denote our winnings. | | | | X can take values -2, -1, 0, 1 , 2, 4. pmf of X:   |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | | O | 0 | 1 | 2 | 0 | 1 | 0 | | B | 0 | 0 | 0 | 1 | 1 | 2 | | W | 2 | 1 | 0 | 1 | 0 | 0 | | X = x | -2 | -1 | 0 | 1 | 2 | 4 | | P(X = x) | = | = | = | = | = | = | | | | | | | | | | | | | | | | | | | | | | |
| Let X = diff btw num of heads and tails when coin tossed n times. What are the possible values of X. If coin is fair, what is the pmf of X? | | | | | | | | | | | | | | | X = nH - nT = (n - nT) - nT = n - 2nT, where 0 ≤ nT ≤ n.  If coin is fair, P(X = n-2i) = P(nT = i) = for 0 ≤ i ≤ n | | | | | | | | | | |
| 5 distinct nums are randomly distributed to players numbered 1 to 5. When 2 players compare their nums, one with higher num wins. Player 1 and 2 compare; winner compare with player 3 and so on. Let X = num of times 1 is winner. Find P(X = i), i = 0,1,2,3,4 | | | | | | | P(X = 0) = P(1 loses to 2) = 1/2. (of the 2 cards btw 1 and 2, 1 has the smaller one)  P(X = 1) = P(of 1,2,3: 3 has largest, then 1, then 2) = =  P(X = 2) = P(of 1,2,3,4: 4 has largest, 1 has next largest) = =  P(X = 3) = P(of 1,2,3,4,5: 5 has largest then 1) = = . P(X = 4) = P(1 has largest) = = | | | | | | | | | | | | | | | | | | |
| F(x) = | | | | P(X = 1) = P(X ≤ 1) - P(X < 1) = 1/2 - 1/4  P(X = 2) = P(X ≤ 2) - P(X < 2) = 11/12 - [1/2 + 1/4] = 1/6  P(X = 3) = P(X ≤ 3) - P(X < 3) = 1 - 11/12 = 1/12  P(1/2 < X < 3/2) = P(X < 3/2) - P(X < 1/2) = [1/2 + 1/8] - 1/8 = 1/2 | | | | | | | | | | | | | | | | | | | | | |
| 4 buses carrying total of 148 students arrives. Buses carry 40,33,25,50 students. One of the student is randomly selected. Let X = num of students on bus carrying randomly selected student. 1 of the 4 bus drivers also randomly selected. Let Y = num of students on selected bus driver bus. | | | | | | | | | | | | | E(X) or E(Y) is bigger? Prob of selecting student on bus is proportional to num of students on that bus but prob a selecting bus driver is 1/4. So E(X) should be larger)  Calculate. P(X = i) = i/148 for i = 40,33,25,50. P(Y = i) = 1/4  E(X) = = [402 + 332 + 252 + 502]/148 ≈ 39.28  E(y) = [40+33+25+50]/4 = 37 | | | | | | | | | | | | |
| Sample of 3 items selected at random from box containing 20 items of which 4 is defective. Find expected num of defective items in sample. | | | | | | | num of defective items D is a hypergeometric dist with (n,N,m) = (3,20,4). E(D) = nm/N = 3/5  OR = 0 + 1 + 2 + 3 = 3/5 | | | | | | | | | | | | | | | | | | |
| Newsboy purchase newspapers at 10cents and sell at 15. He is not allowed to return unsold papers. If daily demand is binomial r.v. w n = 10, p = 1/3. How many papers should he purchase to maximise expected profit?  OR just purchase num as close as possible to expected num sold = 10(1/3) = 3 | | | | Let m be num of copies ordered. 1 ≤ m ≤ 10.  Let X be r.v. for demand and Gm be r.v. of profit when his order is m copies. (k is num sold)  E(Gm) = = 15 = 15  For 1 ≤ m ≤ 9, E(Gm+1) - E(Gm) = {15} – {15} = 15(m+1)P(X = m+1) - 15(m+1)[P(X ≤ m) + P(X = m+1)] + 15mP(X ≤ m) + 5 = 5 - 15P(X ≤ m)  Thus E(Gm+1) - E(Gm) iff P(X , m) ≤ 1/3   |  |  |  |  |  | | --- | --- | --- | --- | --- | | m | 0 | 1 | 2 | 3 | | P(X = m) | 0.017342 | 0.086708 | 0.195092 | 0.260123 | | P(X ≤ m) | 0.017342 | 0.104049 | 0.299141 | 0.559264 |   So optimal order is 3 copies. | | | | | | | | | | | | | | | | | | | | | |
| Let X be r.v. taking values 1 and -1 with P(X = 1) = p = 1 - P(X = -1). Find c ≠ 1 s.t. E[cX] = 1 | | | | | | | | E(cX) = c1P(X=1) + c-1P(X = -1) = cp + (1-p)/c = . Since E(cX) = 1. c2p - c + (1-p) = 0. c = (1-p)/p | | | | | | | | | | | | | | | | | |
| Suppose 4 fair dice are rolled. Let M = min of 4 numbers. What are the possible values of M? Find E(M) | | | | Find P(M ≥ k), where k = 1,2,..., 6. Let Xi = num on die i. P(M ≥ k) = P(X1 ≥ k)P(X2 ≥ k)P(X3 ≥ k)P(X4 ≥ k) (by independence) = =  E(M) = = = = 1.755 | | | | | | | | | | | | | | | | | | | | | |
| Let X~Geo(p), P(X = k) = pqk-1. Show P(X > n) = qn for n ≥ 1 and P(X > n+k|X > n) = P(X > k) for all n,k ≥ 1 | | | | | | | | | P(X > n) = first n trials all failures = qn  P(X > n+k|X > n) = = = qk = P(X > k) | | | | | | | | | | | | | | | | |
| Ball is drawn from urn containing 3W, 3B balls with replacement. | | | | P(of 1st 4 balls drawn, exactly 2 are W) = (3/6)2(3/6)2 = 3/8 | | | | | | | | | | | | | | | | | | | | | |
| Suppose airplane engines will fail w prob 1-p. If airplane needs majority of engines to operate to fly, for what values of p is a 5-engine plane preferable to 3-engine plane? | | | | | | | | | | | | | For 3-engine: P(X ≥ 2) = 3p2(1-p) + p3 = p2(3-2p)  For 5-engine: P(X ≥ 3) = p3(1-p)2 + p4(1-p) + p5 = p3(10-15p+6p2)  We need p3(10-15p+6p2) > p2(3-2p)....simplify... p > 1/2 | | | | | | | | | | | | |
| Suppose biased coin lands on heads w prob p is flipped 10 times. Given that there are 6 heads. Find conditional p that first 3 outcomes are h,t,t. Find conditional p that first 3 outcomes are t,h,t. | | | | | | | | P(h,t,t|6 heads) = = = 1/10  P(t,h,t|6 heads) = = = 1/10 | | | | | | | | | | | | | | | | | |
| Integer N selected at random from {1,2,..., 103}. What happens when 103 replaced by 10k as k ∞ | | | | P(N divisible by 3) = 333/1000 1/3  P(N divisible by 7) = 142/1000 1/7  P(N divisible by 15) = 66/1000 1/15 | | | | | | | | | | | | | | | | | | | | | |
| Roulette (0, 00: green; 1,2,...36: 1/2 red, 1/2 black). Gambler bet $1 on red. If red appears, take $1 profit and quit. If lost, make additional $1 bet on red for next 2 spins and then quit. Let X = winnings | | | | | | | | | | | | | | | P(X > 0) = 18/38 + 20/38 \* 18/38 \* 18/38 ≈ 0.5918  E(X) = 1(0.5918) - 1(2\*20/38\*20/38\*18/38) -3(20/38\*20/38\*20/38) ≈-0.108. So strategy is bad | | | | | | | | | | |
| You have $1000 and a good sells for $2 per ounce. After 1 week, the good will either be $1 or $4 an ounce, with equal prob. | | | | | Maximise expected amt of money at end of week: Buy 500 ounces now and then sell. E(money) = (1/2)(500) + (1/2)(2000) = 1250  Maximise expected amt of good at end of week: Buy after 1 week. E(amt) = (1/2)(1000) + (1/2)(250) = 625 | | | | | | | | | | | | | | | | | | | | |
| $A must be paid if event E occur with prob p. How much to charge customer so that expected profit is 10% of A? | | | | | | | | | | | Let C = amt charged to customer. E(profit) = C - [Ap + 0(1-p)] = C - Ap  For E(profit) = A/10, C-Ap = A/10. C = A(p + 1/10) | | | | | | | | | | | | | | |
| E(X) = 1 and Var(X) = 5. | | | | E[(2+X)2] = Var(2+X) + [E(2+X)]2 = Var(X) + [2+E(X)]2 = 14. Var(4+3X) = 9Var(X) = 45 | | | | | | | | | | | | | | | | | | | | | |
| 4-sided fair die numbered 1,2,3,4 tossed 2 times. Let X = 1st num, Y = 2nd num | | | | P(X - Y = 0) = P(X = y) = 4/16 = 1/4. E(X - Y) = 0  E[(X-Y)2] = (-3)2(1/16) + (-2)2(2/16) + (-1)2(3/16) + 12(3/16) + 22(2/16) + 32(1/16) = 5/2  Var(X - Y) = 5/2 - 02 = 5/2 | | | | | | | | | | | | | | | | | | | | | |
| Approx 80,000 marriages took place. Estimate prob that for at least 1 of these couples  a) both partners born on April 30  b) both partners celebrated bd on same day | | | | | | | | | | | Assuming independence of bd and equal chance of being born on any data,  P(both born on April 30) = 1/3652. Let X = num of couples born on this date. X ~ Binomial(80,000, 1/3652) ≈ Poisson(80000/3652 ≈ 0.6). P(X ≥ 1) = 1 - P(X = 0) = 1 - e-0.6  P(same day) = 1/365. Y ~ ≈ Poisson(80000/365 ≈ 219.18. P(Y . 1) = 1 - e-219.18 ≈ 1 | | | | | | | | | | | | | | |
| Suppose average num of typos on page document is 1. P(no typos on a page). P(no typos in 5-page document) | | | | | | | | | | Let Z = num of typos on page. Z ~ Poisson(1). P(Z = 0)  Let Y = num of typos on 5 pages. Y ~ Poisson(5). P(Y = 0) = e-5 OR P(Y = 0) = P(Z = 0)5 = (e-1)5 = e-5 | | | | | | | | | | | | | | | |
| Num of times person contracts cold in a year is a Poisson r.v. with = 5. Suppose a new drug is marketed as reducing to 3 for 75% of the pop. For the other 25%, no effect. If an individual tries the drug for a year and has 2 colds, how likely is it that the drug is beneficial for him? | | | | | | | | | | | | | | | | | P(beneficial|2 cold) = = ≈ 0.8886 | | | | | | | | |
| P(full house) ≈ 0.0014. Find an approximation for prob that in 1000 hands of poker you will be dealt at least 2 full houses. | | | | | | | | | | | | | | Use poisson approx to binomial. X ~ Poisson(1000\*0.0014 = 1.4). P(X ≥ 2) = 1 - P(X = 0) - P(X = 1) = 1 - e-1.4 - e-1.4(1.4) ≈ 0.4082 | | | | | | | | | | | |
| There are 3 highways. Num of daily accident on these highways are Poisson r.v. w 0.3, 0.5 and 0.7. Find expected num of accidents that will happend on any of these highways today. | | | | | | | | | | | | | | | | | | | | | Let Xi = num of accidents on highway i. E[X1 + X2 + X3] = E[X1] + E[X2] + E[X3] = .3 + .5 + .7 = 1.5 | | | | |
| Suppose 10 balls are put into 5 boxes, w ea ball indep being put in box i w prob pi. Find expected num of boxes that do not have any balls.  Find expected num of boxes that have exactly 1 ball. | | | | | | | | | Let Xi = 1 if box i don't have any balls, 0 otherwise. Then E = = = . (ball not in ith box)10  Let Yi = 1 if box i have exactly 1 ball and 0 otherwise. Then E = = = . (10 balls, only 1 in box i) | | | | | | | | | | | | | | | | |